VOLUME 78 SEPARATE No. 142

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

AUGUST, 1952

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

UNCONFINED GROUND-WATER FLOW TO MULTIPLE WELLS

By Vaughn E. Hansen, J. M. ASCE

IRRIGATION DIVISION

Headquarters of the Society
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Proceedings—Vol. 78 August, 1952 Separate No. 142

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

PAPERS

UNCONFINED GROUND-WATER FLOW TO MULTIPLE WELLS

By Vaughn E. Hansen,[1] J. M. ASCE

Synopsis

The purpose of this paper is to clarify the nature of unconfined flow to single and multiple wells, and to present a method of solving problems associated with this type of flow. The effect of the capillary fringe on the location of the free surface and the form of the flow patterns, the zone of validity of the Dupuit equation, the shape of the free surface near the well, and the variation in the stream surface spacing are all discussed. A functional relationship independent of the radius of influence is established, relating the variables at the well; this relationship applies to both single and multiple wells. A fundamental dimensionless parameter consisting of a ratio of Froude's to Reynolds numbers is formulated that characterizes the shape of the cone of depression around a well. The concepts of well efficiency and effectiveness are clarified and guides are presented for their correct use.

Introduction

Notation.—The letter symbols introduced in this paper are defined where they first appear and are assembled alphabetically in the Appendix for convenience of reference.

The need for a clearer understanding of the flow to unconfined wells and for better methods of solving the problems arising from this flow has arisen as a result of extensive pumping of ground water and the demand for a more economical use of the water. The investigation reported in this paper was undertaken in an effort to contribute further knowledge to both the nature of the flow and the method of solution.

Flow into wells can be divided into two broad classifications, depending upon the boundary conditions. The first classification is confined flow, in

Note.—Written comments are invited for publication; the last discussion should be submitted by **February 1, 1953.**

[1] Irrig. Engr. Irrig. Div., SCS; and Asst. Prof. of Research, Utah State Agri. College, Logan, Utah.

which the water is restricted under pressure between two rather impermeable layers. The second classification is unconfined flow, the upper surface of which is at atmospheric pressure.

Since confined flow occurs between fixed boundaries, it is susceptible to rather complete analysis based on relatively few approximations and assumptions, with the result that the theoretical solutions conform very closely to the observed conditions. Consequently, considerable knowledge is available regarding the theory and practical aspects of such flow.

Unconfined flow is much more involved, however, because of the existence of an unbounded water surface. Since the shape of the surface is not known, an approximate analysis based on extensive assumptions must be made, thus imposing rather severe limitations on the resulting solution. The general acceptance of the solution, without understanding the inherent limitations, has produced a certain degree of complacency among many practical ground-water engineers, as well as a tendency to disregard any observations at variance with the expected solution. It is important, also, that the theoretician clearly appraise the limitations imposed by nature on the too extensive application of equations of ground-water flow; nature rarely provides the idealized case assumed in most theoretical developments. However, the simplified situation must remain the basis of study, with the knowledge that it does not occur, and the results must be modified accordingly.

In this study, the usual assumptions of steady flow through a homogeneous porous medium have been made. For the confined flow, the bounding surfaces are assumed to be parallel, horizontal, impermeable layers; for the unconfined flow, the wells are assumed to penetrate completely the homogeneous medium overlying an impermeable, horizontal bed.

General Flow Analysis

A general analysis of flow through porous media has been extensively covered by several writers.[2,3] The essentials of these analyses, as applied to the case of steady flow into wells, will be summarized in order to interpret the significance of the experimental results.

Darcy's law is

$$v = K \frac{dh}{ds} \qquad (1)$$

in which v is velocity, K is the coefficient of permeability, h is the piezometric head, and s is the length along a stream line. When Eq. 1 is combined with partial differential equation of continuity $\nabla v = 0$ for confined flow, the Laplace equation ($\nabla^2 h = 0$) is obtained. When the boundary conditions for confined radial flow are applied and the differential equation is integrated, the following equation for the shape of the piezometric surface of a single artesian well is obtained

$$h - h_1 = \frac{Q}{2 \pi K t} \log_e \frac{r}{r_1} \qquad (2)$$

[2] "Flow of Ground Water," by C. E. Jacob, "Engineering Hydraulics," John Wiley & Sons, Inc., New York, N. Y., 1950, Chapter 5.

[3] "The Flow of Homogeneous Fluids Through Porus Media," by M. Muskat, J. W. Edwards, Inc., Ann Arbor, Mich., 1946.

in which Q is the total discharge, t is the thickness, and r is the radius of the well. Since the Laplace equation is linear in h, the solution for multiple wells is simply the sum of the solutions for each individual well.

As was previously stated, the existence of a free surface greatly complicates the analysis. One writer has gone so far as to claim that the problem cannot be solved with the present knowledge of mathematics. This, of course, is an extreme statement of the complexity of the problem, for through the use of simplifying assumptions, clearly understood, a solution may be obtained that will be of considerable value. The failure to understand these assumptions and their effect upon the solution has been the root of a great deal of confusion in the minds of many practical as well as theoretical investigators.

The classical analysis of this problem was presented by Jules Dupuit in 1863. He assumed horizontal flow throughout a homogeneous material underlain by an impermeable stratum with a well completely penetrating the permeable material. Necessarily, then, the flow occurred through concentric cylinders of a variable height h that were potential surfaces, the last and smallest cylinder having a height equal to the depth of the water in the well. The result of an analysis based on these assumptions is the formula commonly known as the Dupuit equation:

$$h^2 - h^2_1 = \frac{Q}{\pi K} \log_e \frac{r}{r_1} \qquad (3)$$

Limitations of the Dupuit Equation.—Since the free surface approaches the horizontal at a considerable distance from the well, it can be seen that the assumption of horizontal flow through vertical potential surfaces will become more accurate as the radius increases, with a resulting increase in the accuracy of the Dupuit equation.

The surface given by the Dupuit equation intersects the well at the level of the water in the well, whereas it is a known fact that the free surface intersects the well above this point, giving rise to what is commonly referred to as a zone of seepage that increases in extent as the drawdown increases. Moreover, as the fluid approaches the well, the free-surface slope increases, causing the potential surfaces (that must be normal to the flow surfaces) to depart more and more from the cylindrical form assumed in the derivation. For these reasons the actual free surface would be above that calculated by the Dupuit equation, the difference being greatest at the edge of the well.

Effect of a Capillary Zone.—In addition to the flow conditions already mentioned, the effect of the capillary zone must be considered. This is especially true for model studies wherein the capillary rise may be an appreciable percentage of the depth of flow. Ordinarily it is sufficiently accurate to say that the free surface is the surface below which the voids are saturated with fluid. However, when a capillary zone becomes important because of its relative size, this definition is no longer sufficient and the free surface can best be defined as the surface of atmospheric pressure. R. D. Wyckoff, H. G. Botset, and Mr. Muskat,[4] and others have found that the potential distribution extends

[4] "Flow of Liquids Through Porous Media Under the Action of Gravity," by R. D. Wyckoff, H. G. Botset, and M. Muskat, *Physics*, Vol. 3, 1932 pp. 90–114.

across the atmospheric-pressure surface into the capillary zone. This potential distribution in the capillary zone will give rise to additional flow that cannot be neglected.

Experimental Studies

Several investigators, recognizing the limitations of the Dupuit equation, have conducted model tests to ascertain closely the true nature of the complex problem of unconfined flow. In 1932, Messrs. Wyckoff, Botset, and Muskat[4] constructed a 15° sand sector with glass sides and piezometers connected to the bottom. They found that the base piezometric heads, rather than the free surface, were given by the Dupuit equation. By injecting dye into the sand at the inflow face and tracing its movement to the well, they found that the capillary zone contributed a sizeable discharge. These measurements were used to substantiate a modification of the Dupuit equation that allows for the increased flow resulting from capillarity. Apparently no attempt was made to determine the effect of the capillary flow on the free surface.

Harold E. Babbitt, M. ASCE, and David H. Caldwell,[5] A.M. ASCE, have undertaken a very extensive three-phase study of unconfined flow. The first phase of the study was a series of electrical-analogy tests on a thin carbon wedge representing a single well. In addition to verifying the previously mentioned limitations of the Dupuit equation, they developed an equation for the free surface near the well.

The second phase of the investigation was the construction of a 15° sand sector 100 in. long and 16 in. high having a single well located at the apex. Along one of the sides the piezometric head was measured and the free surface determined. The discharges were compared with those given by the equation developed by the electrical-analogy method and were found to be in good agreement.

A study of multiple wells comprised the third phase. A 13-ft square container was filled to a depth of 13 in. with ordinary building sand. Water was admitted through screens located at the corners and flowed toward various combinations of 5 wells placed near the center. The discharges were compared with those computed from a multiple-well analysis based on certain simplifying assumptions. A fair agreement was secured.

Since the study by Messrs. Babbitt and Caldwell represents the most exhaustive investigation of unconfined flow known to the writer, it has been used as the starting point for the experiments reported herein. For that reason an appraisal of the results of the Babbitt and Caldwell tests is needed at this point to understand clearly the objectives of this investigation.

First of all, with regard to the electrical-analogy study, it is felt that the resulting free surface is a good representation of the true free surface. However, caution should be used near the well, since the wedge was very thin and any small cracks or lack of homogeneity as well as improper electrode contact would be more noticeable in this region. Moreover, the range of variables

[5] "The Free Surface Around, and Interference Between, Gravity Wells," by Harold E. Babbitt and David H. Caldwell, *Bulletin Series No. 374*, Eng. Experiment Station, Urbana, Ill., January 7, 1948.

studied was rather small. For example, the maximum external radius was only 25 well radii and varied from 3.2 to 6.5 times the maximum depth of flow.

The formula proposed by Messrs. Babbitt and Caldwell was presented in the form

$$h_e - h = \frac{2.3\,Q\,C_x}{\pi\,K\,h_e} \log \frac{r_e}{0.1\,h_e} \qquad (4)$$

in which the subscript e refers to the external boundary, and the coefficient C_x is defined by a curve relating the drawdown to the radius. Of extreme interest is the fact that when C_x is plotted against the logarithm of the quantity (r/r_e), a straight line results to a value of $r/r_e = 0.05$. Since the minimum value of r/r_e in the electrical-analogy experiments was only 0.04 and the maximum 0.087, the logarithmic relation holds essentially true to the edge of the well.

When the approximation ($C_x = 0.3 \log r/r_e$) is substituted in Eq. 4, the following formula results:

$$h_e - h = \frac{0.69\,Q}{\pi\,K\,h_e} \log \frac{r_e}{0.1\,h_e} \log \frac{r_e}{r} \qquad (5)$$

Noteworthy is the fact that the elevation of the free surface is expressed as a linear function of the radius in a manner very similar to that for artesian wells, whereas the Dupuit equation is not a linear function.

Model Studies of Unconfined Flow

Equipment for Model Studies.—The sand-model studies described herein were undertaken at the laboratory of the Iowa Institute of Hydraulic Research, principally for the purpose of gaining further information about the free surface near the well. A 90° sector containing sand (Fig. 1) was constructed, in which the inflow screen was 120 in. from a central well having a radius of 1.2 in. The depth of inflow was maintained at approximately 24 in. in 30 in. of sand. Two other wells, each of 1.2-in. radius, were located 36 in. from the apex. The wells were placed along planes of symmetry so that by the principle of images the full pattern of wells would be in the center of a circular confining boundary. One of the planes of symmetry was made of plexiglas, containing 174 piezometer connections. The inflow face was formed of two layers of copper screen supported by 1- × 4-in. timber ribs. The base of the model was constructed of concrete and the sides, exclusive of the plexiglas, were of ½-in. plywood. Two layers of copper screen, soldered to ½-in. hardware cloth, formed the well face. Water entered the sand through the screen at the outer periphery and flowed toward the wells, at which point the water surface was maintained at the required level by an overflow control. The discharge was measured by weighing the water flowing from the wells.

The sand used in the model was graded Iowa River mason's sand, with most of the fines removed by washing. To reduce the amount of trapped air, which has a considerable and variable effect upon the permeability,[6] the sand was placed under water. Each layer was approximately 1-in. thick and raked

[6] "Effect of Entrapped Air upon the Permeability of Soils," by J. E. Christiansen, *Soil Science*, Vol. 58, No. 5, November, 1944.

thoroughly with a leaf rake to free any additional air. Moreover, the raking reduced the horizontal stratification as well as the orientation of the grains.

Procedure for Model Studies.—The sequence of operations in the series of tests was determined by the criterion that the water level in the sand should

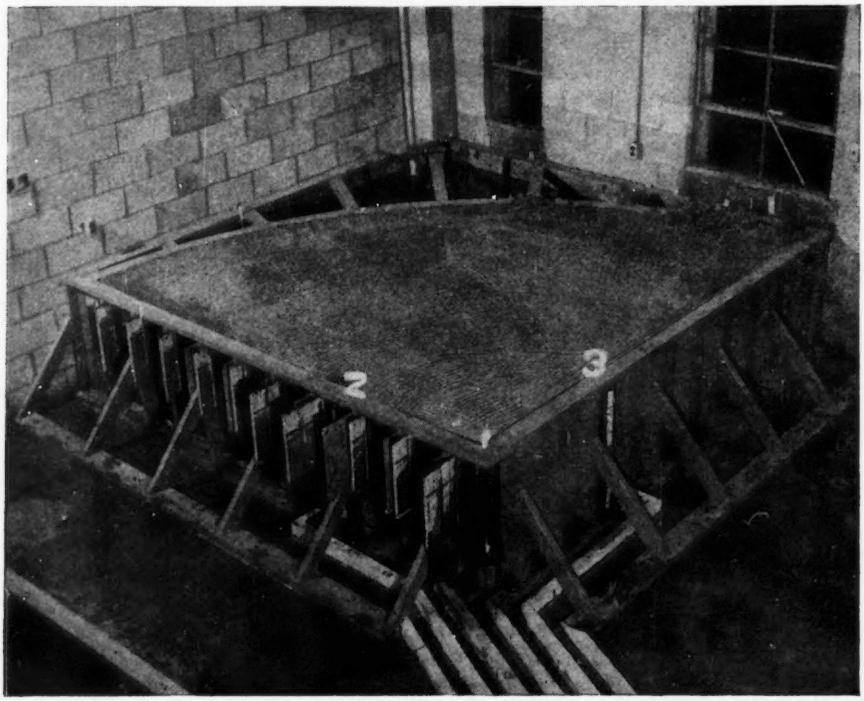

FIG. 1.—PHOTOGRAPH OF SAND MODEL

not be raised after it was once lowered. By raising the level, even though slowly, air would be trapped in the pores and the permeability would thus be reduced. This effect could very well be serious near the well at which point the drawdown is a maximum.

The water surface in the central well was lowered to 18.3 in. and the reservoir level to approximately 24 in., which was maintained during the remainder of the tests. Since the sand was 30 in. deep, this differential allowed for the expected maximum capillary rise of 6 in. to be completed near the outer periphery. The piezometric heads, together with the discharge, were used in determining when an equilibrium condition was approached.

Triple-strength red food coloring was placed in all piezometer tubes. This dye served two very useful purposes in addition to functioning as the piezometer fluid. The dye could be made to enter the flow and trace out the stream lines (Fig. 2) by either placing additional dye in the tubes or simply compressing the rubber tube leading to the piezometer connection. Moreover, when the

tubes were given an initial compression, the velocity of the resulting dye front could be determined. The piezometer readings were corrected for the increase in specific gravity resulting from the coloring.

Experimental Tests on Model.—Seven complete tests were made. The central well, referred to hereafter as well 1 (Figs. 1 and 3), was operated at four

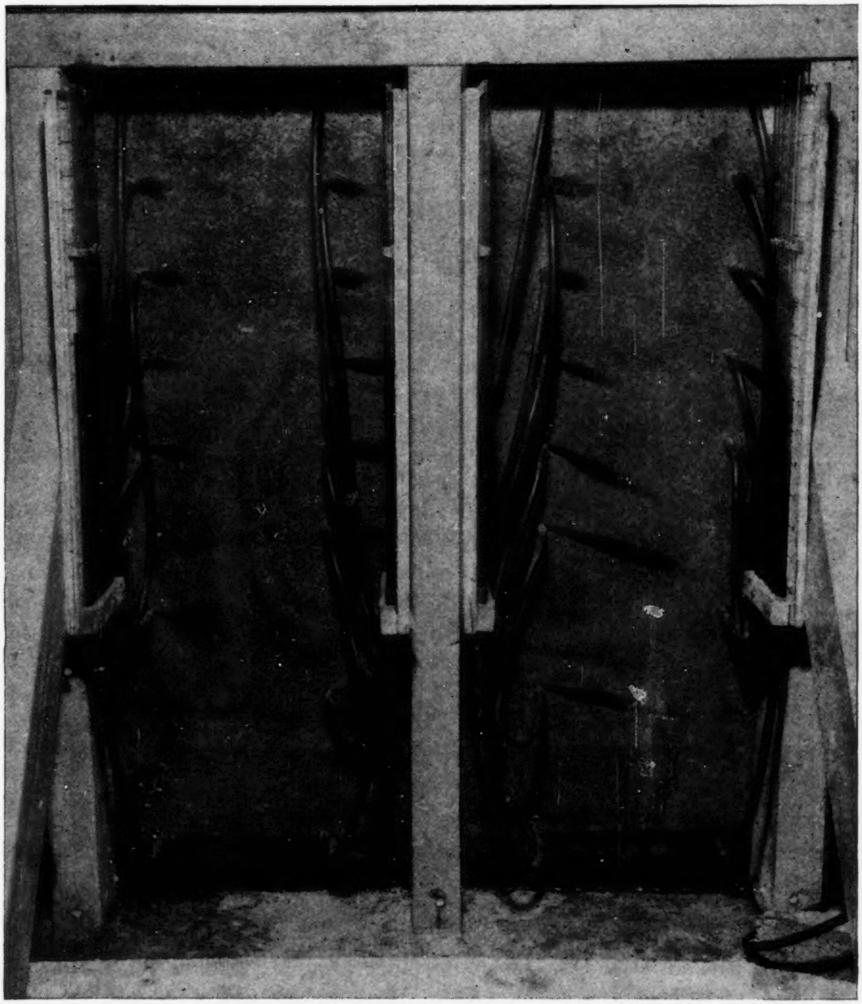

FIG. 2.—DYE FRONTS TRACING OUT STREAM LINES

levels: 18.3 in., 12.3 in., 6.3 in., and complete drawdown. (All elevations were measured above the base of the model.) A typical result is shown in Fig. 4.

The fifth test set up consisted of 3 wells in a straight line with the central well maintained at full drawdown and the outside wells at 12.9 in. elevation.

The potential distribution for this case was obtained in two steps. In the first step well 1 was at full drawdown and well 3 at 12.9 in. The flow pattern thus determined along the plexiglas boundary was then at right angles to the line of wells. The second step was to close well 3 completely and lower the water sur-

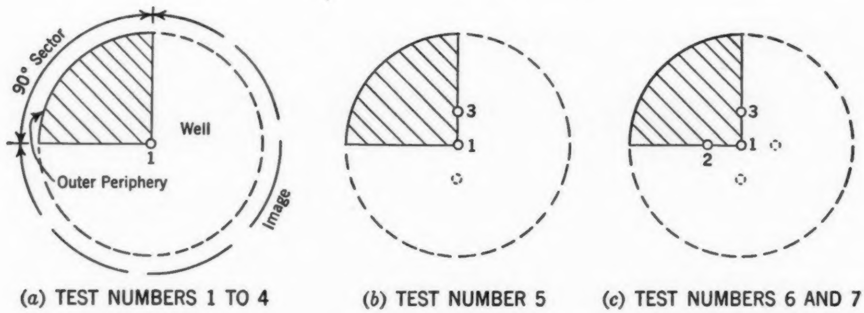

FIG. 3.—PLAN OF WELL LAYOUT FOR MODEL TESTS

face in well 2 to 12.9 in. In this manner the flow pattern along the line of wells was determined. It will be observed that it was necessary to raise the water surface over a portion of the flow area, but the tests were arranged so that this fluctuation always occurred at well 3, the farthest from the line of piezometers. In this way it was hoped to minimize the effect of trapped air.

The sixth test was made with 4 wells in a square and the fifth well in the center. The pattern consisted of 3 of the wells in a line along a diagonal operat-

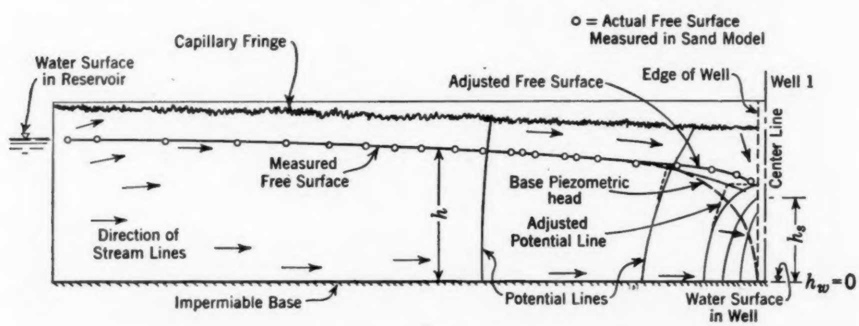

FIG. 4.—FLOW PATTERN FOR TEST NUMBER 4

ing at full drawdown, and the remaining 2 operating at 12.9 in. The first stage was obtained with wells 1 and 3 at full drawdown and well 2 at 12.9 in. The flow pattern thus obtained was at right angles to the line of 3 wells at full drawdown. When well 2 was lowered to full drawdown and well 3 was raised to 12.9 in., the flow pattern along the line of the 3 wells was obtained.

The last test, number seven, was with all 3 wells at full drawdown, resulting in a pattern of 5 wells under full operation; the results of this test are shown in Fig. 5.

Analysis of Test Results.—The discussion of the sand-model tests will be separated into four categories—the Dupuit analysis, the capillary zone, the free surface, and the boundary conditions at the well.

The Dupuit Analysis.—The Dupuit equation has been shown previously to agree with the piezometric heads measured along the bottom of the permeable

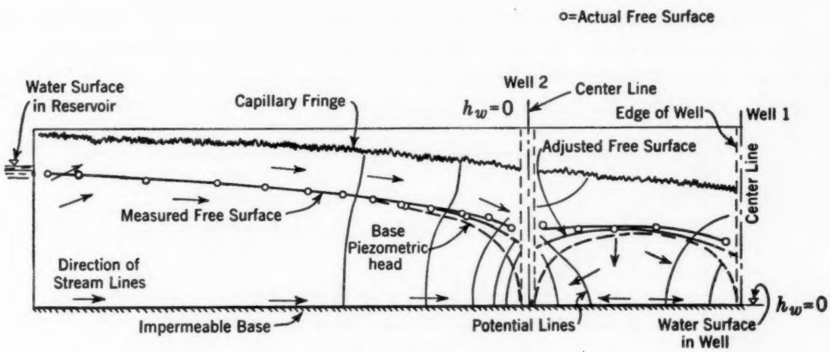

FIG. 5.—FLOW PATTERN FOR TEST NUMBER 7

stratum. Additional experimental verification is shown in Fig. 6, in which the Dupuit equation is the best fit curve for the experimental data obtained from tests one to four. With the experimental tests indicating that the Dupuit equation accurately gives the free surface at considerable distances from the well and also the piezometric head along the entire base, a re-examination of the Dupuit analysis will be of interest in ascertaining the reason for agreement in these regions. The basic assumption of horizontal flow through vertical potential surfaces is essentially met at considerable distances from the well. Furthermore, since the potential surfaces are normal to the horizontal impervious bed, the assumption of horizontal flow through vertical potential surfaces is also reasonable in this region. The flow between the base and an adjacent flow surface is

$$q = 2\,\pi\,r\,\Delta n\,K\,\frac{dh}{dr} \qquad (6)$$

in which q is the discharge between flow surfaces. When it is assumed that $\Delta n = C\,h$ and therefore $q = C\,Q$, the expression may be integrated to obtain the Dupuit equation. Since the resulting equation is verified experimentally, the assumption is evidently sound, and can therefore be used in further defining the pattern of flow. In brief, this analysis means that the normal distance between stream surfaces immediately adjacent to the impermeable base remains a constant fraction of the piezometric head above the impervious base as the fluid approaches the well.

Effect of the Capillary Zone.—The capillary zone had considerable effect upon the results, as its relative magnitude would indicate—the inflow depth being approximately 24 in. and the average capillary rise being 5 in. From Figs. 4 and 5, the potentials can be seen to extend across the free surface and

well into the capillary zone. This, of course, should be expected since the flow of the fluid depends on the gradient and not the absolute magnitude of the piezometric head. The potential pattern is modified to some extent, however, by the difference in permeability that occurs in the fringe itself. In any graded granular material, a fringe rather than a definite clear-cut line will exist at the top of the capillary zone.

The Free Surface.—The inflow surface that must exist in all model studies will be considered first. When the fluid enters the permeable material, the capillary forces immediately begin to affect the flow pattern. The fluid is drawn up to a height equal to the capillary rise above the free surface of the

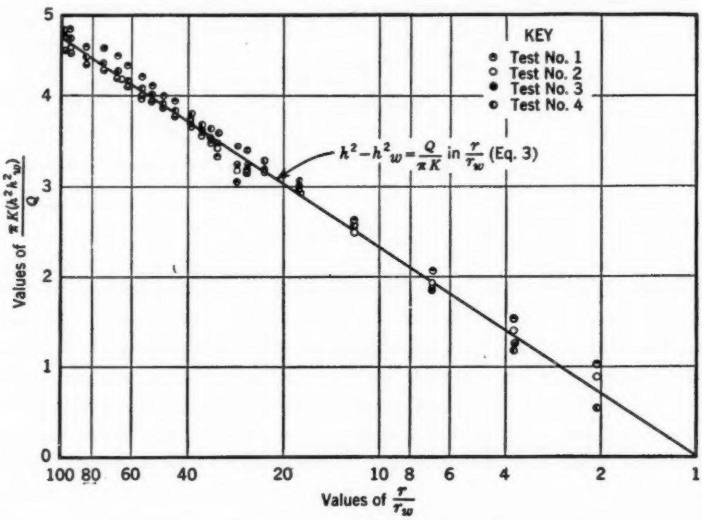

FIG. 6.—BASE PIEZOMETRIC HEAD FOR UNCONFINED FLOW

water in the sand. This rise is completed in a relatively short distance. Since the flow surfaces and potential surfaces must be orthogonal, the potential surfaces will be moved rearward by the upward flow to the capillary zone. Hence, at a given point, the potential,

$$h = p/\gamma + z \qquad (7)$$

(in which γ is the weight per unit volume and z is the elevation) will decrease, and since the elevation z remains constant, the pressure p must decrease. Consequently, the atmospheric-pressure surface, or what is hereafter referred to as the free surface, is lowered slightly as a result of this upward flow. Once the capillary zone is fully developed, the location of the free surface is not affected until the flow reaches the vicinity of the well. Throughout the central portion of the flow, the free surface is a stream surface above which the fluid within the capillary zone is flowing at less than atmospheric pressure. In order for the fluid to enter the well that is at atmospheric pressure, the pressure in the fluid must be increased until it is equal or above atmospheric pressure by a decrease

in elevation. The resulting flow across the atmospheric-pressure line near the well causes the potential surfaces to lower in order that the potential and stream surfaces remain normal. Hence, at a given point the potential ($h = p/\gamma + z$) is increased by an increase in pressure, resulting in a higher atmospheric-pressure line. Thus, the capillary flow has lowered the free surface when the flow is established at the periphery, and raised the free surface in the vicinity of the well, as can be seen from Figs. 4 and 5.

The flow-net principles were used to estimate the extent to which the free surface was altered by the downward capillary flow near the well. Since the free surface unaffected by capillary flow is a stream surface that must be normal to each potential surface, the atmospheric-pressure surfaces measured in the sand-model tests were adjusted until the orthogonality requirement was met, as shown in Fig. 4. The correction was considerable and could not be ignored. When the corrected free surface was plotted against the logarithm of the term

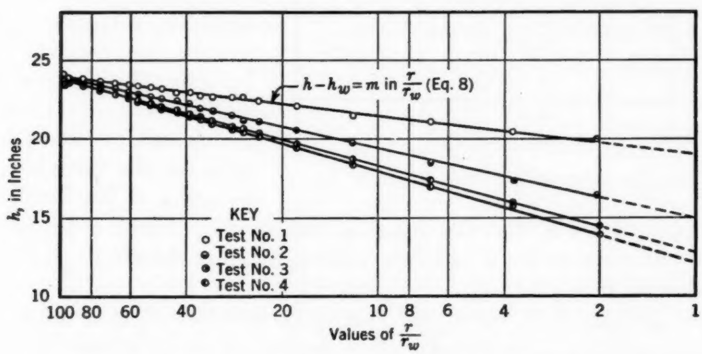

FIG. 7.—FREE SURFACE AS A FUNCTION OF RADIUS

r/r_w (Fig. 7), an essentially straight line was obtained, giving a linear logarithmic function of the form

$$h - h_1 = m \log_e \frac{r}{r_1} \qquad (8)$$

in which m is the slope and has the dimension of a length.

Since a similar equation was obtained by the writer from a re-examination of the Babbitt-Caldwell data, sufficient experimental confirmation of the linear logarithmic nature of the free surface near the well is available to justify an analysis of the flow pattern to learn the implication of such a relation. Consider the differential flow ($q = C_1 Q$) occurring between the free surface and an adjacent stream surface. By assuming that the vertical distance between stream surfaces remains constant ($\Delta z = C_2$) as the flow approaches the well, the following linear logarithmic equation is obtained by combining the Darcy and continuity equations, and integrating

$$h - h_1 = \frac{Q\, C_1}{2\, \pi\, K\, C_2} \log_e \frac{r}{r_1} \qquad (9)$$

Data obtained from a sufficient range of the controlling variables are not available to enable the establishment of the parameters involved in the constants C_1 and C_2. Additional data should aid materially in defining these terms. However, from Fig. 8, in which m is plotted against Q for the single well tests, it can be seen that Q is the primary variable. Hence, the assumption that the vertical distance between the free surface and an adjacent flow surface remains constant, not only results in a form of the equation verified experimentally, but also the resulting equation gives a very good clue as to the nature of the coefficient m. Again, by referring to Eq. 5, a further confirmation of the foregoing conclusions regarding the nature of the coefficient is obtained. When additional data for the complete determination of the coefficient m are available, it is felt that the equation will be more useful if the boundary conditions at the well are used as the reference rather than those at the radius of influence, as in Eq. 5.

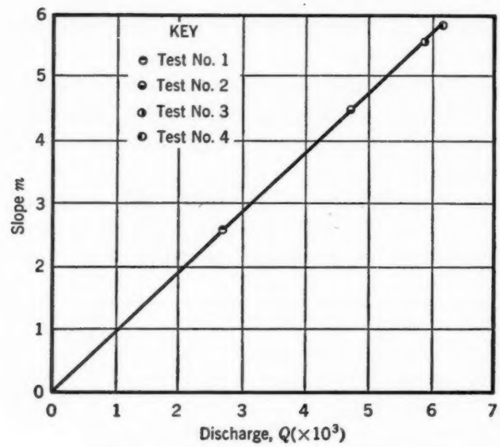

FIG. 8.—RELATION BETWEEN m AND Q

Under ordinary field conditions there is no definite radius of influence, because the conditions far from the well are dependent upon the natural recharge, so an analysis of the variables at the well was undertaken in an attempt to define the flow conditions. When the variables Q, K, r_w, h_w, and h_s at the well are combined to form dimensionless parameters, the following functional relationship is obtained

$$\frac{Q}{K\, r^2_w} = f\left(\frac{h_w}{h_s}, \frac{h_w}{r_w}\right) \dots\dots\dots\dots (10)$$

in which h_s is the height of the intercept of the free surface with the edge of the well.

Boundary Conditions.—Using the results of Messrs. Babbitt and Caldwell's electrical-analogy tests and the writer's data from the sand-model tests, the parameters were plotted as shown in Fig. 9, thereby obtaining the discharge for unconfined well flow in terms of the boundary conditions at the well. Because of the nature of the dimensionless parameters, a family of curves was quite accurately obtained from the available experimental data. However, such a presentation fails to give the values of h_s for the important case of complete drawdown, h_w/h_s being zero for all values of $\frac{Q}{K\, r^2_w}$. For this reason the vari-

ables at the well were grouped into the dimensionless form,

$$\frac{Q}{K\,r^2_w} = f\left(\frac{h_s}{r_w}, \frac{h_w}{r_w}\right) \qquad (11)$$

The resulting family of curves (Fig. 10), although less accurate because of the range of the experimental data, magnify the relationships when $h_w \rightarrow 0$.

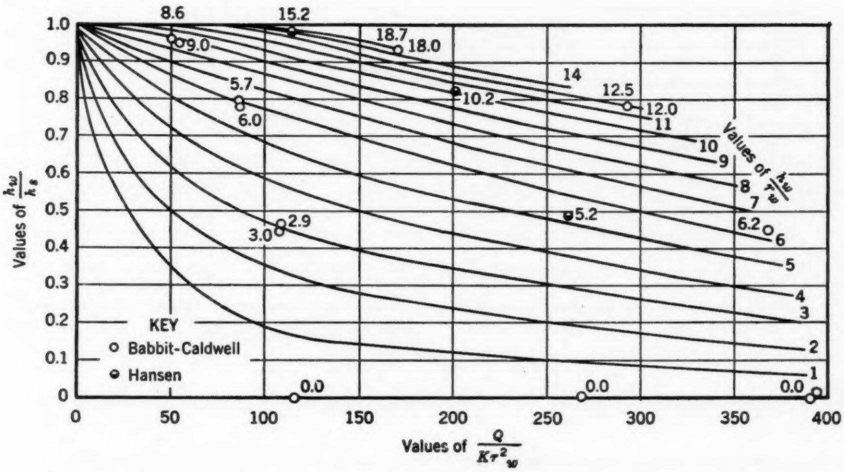

FIG. 9.—UNCONFINED WELL FLOW IN TERMS OF BOUNDARY CONDITIONS AT THE WELL

To verify this functional relation further, the values obtained from the plots were compared with the results for the multiple-well tests and found to be in good agreement. This means that unconfined flow to either single or multiple wells in any arrangement can be defined in terms of the boundary conditions at the well, and hence, the vague quantity—radius of influence—can be avoided.

It should be remembered that both dimensionless plots are based on limited data. Consequently, the results should be taken as a positive indication of the possibilitites of such an analysis, and not as the final answer. When the results of other experimental work are available, the usefulness of the plots can then be extended.

Since the free surface next to the well is not only hard to measure, but is also affected considerably by the entrance conditions, the intercept at the well was obtained by plotting the free-surface elevations against the logarithm of the term r/r_w and extending the essentially straight line to the well (Fig. 7). A similar procedure is recommended when field data are used to relate the boundary conditions at the well.

The fundamental nature of the dimensionless parameter $\frac{Q}{K\,r^2_w}$ should not be overlooked. This parameter is an index of the shape of the cone of depression of the water table around the well, large values being characteristic of the deep cones, small values characterizing shallow depressions.

An analysis will show that this parameter, fundamental in all well flow, is nothing more than the ratio of the Froude and Reynolds numbers, which is the ratio of the viscous to the gravity forces; the inertial forces cancel because of their minor importance in comparison to the viscous and gravity forces.

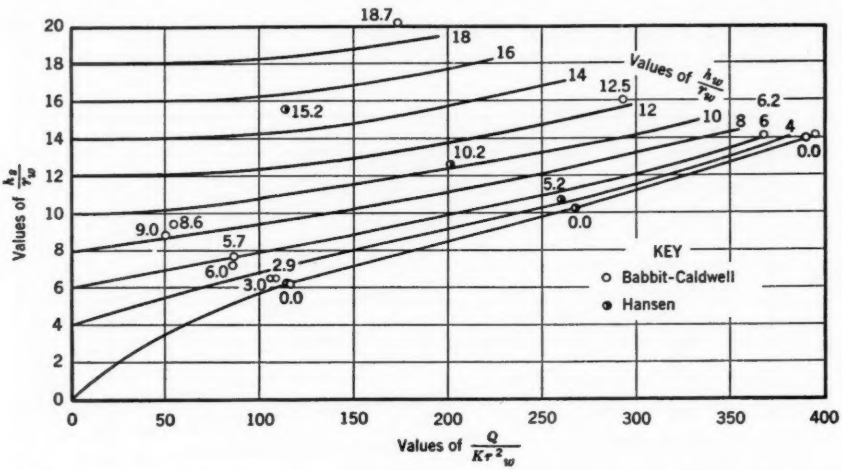

FIG. 10.—UNCONFINED WELL DISCHARGE AS A FUNCTION OF DRAWDOWN AND FREE SURFACE INTERCEPT

When into the basic well parameter $\frac{Q}{K\, r^2_w}$ the coefficient of permeability $K = \frac{C\, d^2\, \gamma}{\mu}$ (in which μ is the viscosity) is substituted, the following results

$$\frac{Q}{K\, r^2_w} = \frac{Q\, \mu}{C\, d^2\, \gamma\, r^2_w} \qquad (12)$$

Considering $C\, d^2$ as a representative area characterizing the porous media through which the flow occurs, the quotient $\frac{Q}{C\, d^2}$ has the dimensions of velocity that could be considered as characteristic of a flow produced by a given boundary condition. Substituting v' for $\frac{Q}{C\, d^2}$, together with $\gamma = \rho\, g$, into the well parameter, the following equation is obtained:

$$\frac{Q}{K\, r^2_w} = \frac{v'\, \mu}{\rho\, g\, r^2_w} \qquad (13)$$

When numerator and denominator are multiplied by v',

$$\frac{Q}{K\, r^2_w} = \frac{\dfrac{(v')^2}{g\, r^2_w}}{\dfrac{\rho\, v'\, r_w}{\mu}} \qquad (14a)$$

or

$$\frac{Q}{K\, r^2_w} = \frac{\tilde{\tilde{F}}}{\tilde{\tilde{R}}} \qquad (14b)$$

in which $\overset{\approx}{F}$ and $\overset{\approx}{R}$ are the Froude and Reynolds numbers, respectively. However, since

$$\overset{\approx}{F} = \frac{\text{inertia forces}}{\text{gravity forces}} \qquad (15a)$$

and

$$\overset{\approx}{R} = \frac{\text{inertia forces}}{\text{viscous forces}} \qquad (15b)$$

then

$$\frac{Q}{K\,r^2_w} = \frac{\overset{\approx}{F}}{\overset{\approx}{R}} = \frac{\text{viscous forces}}{\text{gravity forces}} \qquad (16)$$

Hence, the larger the $\overset{\approx}{F}/\overset{\approx}{R}$ ratio, the greater the viscous forces become with respect to the gravity forces and consequently, the deeper the cone of depression around the well. Since this ratio has as its primary variable the discharge of the well, it is suggested that $\frac{Q}{K\,r^2_w}$ be called the discharge number.

Well Effectiveness.—A common expression in well hydraulics is "well effectiveness," used as a measure of the losses occurring at the well casing. Well effectiveness is generally defined as

$$E_W = \frac{h_e - h_s}{h_e - h_w} \qquad (17)$$

or the ratio between the drawdown measured outside and inside the well.

If the piezometric heads in Eq. 17 are measured in the zone in which Dupuit's equation applies, the results are indicative of the losses occurring at the well casing. However, when the piezometric heads are measured outside the Dupuit region and in particular at or near the free surface, the definition given is not only erroneous but grossly misleading. In fact, from Fig. 9 it is seen that h_w/h_s varies from 0 to 1.0, depending on the values of h_w/r_r and $\frac{Q}{K\,r^2_w}$. When the possible extremes of h_w/h_s of 0 and 1.0 are substituted into Eq. 17, it is seen that the value of E_W can be varied over the entire range from 0 to 1.0 without any loss occurring at the well.

Too often these basic principles are ignored, with the result that the seepage face that must accompany this type of flow is considered as well loss caused by poor design or poor development of the well. When subsequent extensive development fails to alter the flow pattern materially, the premature conclusion is too often reached that either the driller does not know his profession or the formulas are of no value.

If the free surface near the well is observed, the piezometric heads should be plotted against the logarithm of the radius and the resulting straight line extrapolated to its intersection with the well casing. The losses occurring at the well casing can then be determined by comparing the extrapolated value of h_s with the value calculated by the use of Fig. 9 or Fig. 10. The ratio between these two values is a measure of well efficiency. When the square of the piezometric heads plotted against the logarithm of the radius results in a straight line, the

indication is that the measurements were taken in the Dupuit zone. The intersection of the resulting straight line extrapolated to the well casing, when compared to the depth of the water in the well, will also be a measure of well efficiency. The loss occurring at the well will be the difference between these two values.

The loss occurring at the well casing may be the result of one or both of two factors: (1) In the derivation of the well equations, the permeability was assumed to be constant throughout the porous media. If silt and clay are carried by the water to the casing and their presence tends to restrict the flow passages, the permeability will decrease, resulting in an increased loss of energy, and (2) in the derivation, it was assumed that Darcy's law applied; however, Darcy's law applies only when the flow is laminar. If the flow becomes turbulent, the loss of head is more nearly proportional to the second power of the velocity. Hence, the two factors responsible for low well effectiveness or efficiency are decreased permeability and turbulent flow.

To clarify further the fact that the seepage face is not a well loss, the discharges from unconfined and confined wells of similar geometry and location will be compared. When the Dupuit equation for unconfined flow is solved for the discharge,

$$Q_u = \frac{\pi K (h^2 - h^2_w)}{\log_e \dfrac{r}{r_w}} \qquad (18)$$

Likewise, the confined discharge is

$$Q_c = \frac{2 \pi K t (h - h_w)}{\log_e \dfrac{r}{r_w}} \qquad (19)$$

The value of h represents some peripheral boundary condition in each case, so that the depth of water in each well (h_w) in addition to K, r, and r_w are equal; then by dividing the unconfined discharge (Q_u) by the confined discharge (Q_c) an interesting equation is obtained:

$$\frac{Q_u}{Q_c} = \frac{h + h_w}{2 t} \qquad (20)$$

Since t must be less than h_w, which must likewise be less than h ($t < h_w < h$), the unconfined discharge must always be greater than the confined discharge ($Q_u > Q_c$) for comparable boundary conditions. Consequently, the existence of the seepage face does not result in a less efficient well system.

Conclusions

The potential distribution for unconfined flow was found, by model experiments on single and multiple wells, to extend essentially unaltered across the atmospheric-pressure surface and completely through the capillary zone into the capillary fringe. In model studies in which the height of the capillary rise is a significant fraction of the model size, the effect of the capillary flow on the shape of the free surface cannot be ignored. However, when the free surface

for a single well under various drawdowns is corrected for the effect of the capillary flow, the shape of the free surface near the well can be closely approximated by a linear logarithmic function. The free surface in unconfined well flow must always intersect the well casing above the water surface in the well, resulting in a zone of seepage that increases as the drawdown increases. The free surface, consequently, is above that calculated by the Dupuit equation, the difference being greatest at the edge of the well.

Measurements of the base piezometric head show a good agreement with the Dupuit equation; analysis indicates that this equation should not only accurately describe the base piezometric head distribution but also describe the free surface with increasing accuracy as the radius increases. Over the zone in which the Dupuit equation applies, the stream-surface spacing is proportional to the piezometric head. Very near the free surface, at which the linear logarithmic solution applies, the vertical distance between stream surfaces remains essentially constant as the flow approaches the well.

Based upon experimental data, functional relationships (Figs. 9 and 10) between the variables at the well have been developed for unconfined flow. These relationships apply to the solution of either single or multiple systems. The dimensionless parameter $\frac{Q}{K\, r^2_w}$ characterizes the shape of the cone of depression occurring around a well. This parameter, fundamental in all well flow, is nothing more than the ratio of the Froude and Reynolds numbers or the ratio of the viscous to the gravity forces. Since this ratio has as its primary variable the discharge of the well, it is suggested that $\frac{Q}{K\, r^2_w}$ be called the discharge number.

The commonly used term "well effectiveness" is grossly misleading for unconfined flow unless piezometric heads are measured in the zone in which the Dupuit equation applies. When the square of the piezometric heads plotted against the logarithm of the radius results in a straight line, the indications are that the measurements were taken in the Dupuit zone. The intersection of the resulting straight line, extrapolated to the well casing, when compared to the depth of the water in the well, will be a measure of well efficiency. The loss occurring at the well will be the difference between these two values. If the free surface near the well is observed, the piezometric head should be plotted against the logarithm of the radius and the resulting straight line extrapolated until it intersects the well casing. The losses occurring near the well casing can then be determined by comparing the extrapolated value of h_s with that calculated from the curves of Fig. 9 or Fig. 10. The ratio between these two values is also a measure of well efficiency. Well losses can be attributed to either a decrease in permeability near the well or to turbulent flow. The zone of seepage is not a well loss.

Acknowledgments

This paper is based on a dissertation entitled "Evaluation of Unconfined Flow to Multiple Wells by the Membrane Analogy" submitted by the author in partial fulfilment of the requirements for the degree of Doctor of Philosophy, in

the Department of Mechanics and Hydraulics, in the Graduate College of the State University of Iowa, in Iowa City, June, 1949. The research was conducted at the Iowa Institute of Hydraulic Research, with the wholehearted financial assistance of the Department of Mechanics and Hydraulics of the University.

Appreciation for his cooperation is extended to Hunter Rouse, M. ASCE, under whose direction this study was made. To Emmett M. Laursen, A.M. ASCE, the author gives very sincere thanks for his many helpful suggestions.

Appendix. Notation

a = flow area;

C_x = Babbitt-Caldwell dimensionless, variable coefficient relating the drawdown to the radius;

E_W = well effectiveness;

h = piezometric head, $p/\gamma + z$:

- h_e = piezometric head at the radius of influence;
- h_s = height of the intersection of the free surface with the edge of the well;
- h_w = piezometric head in the well;

K = coefficient of permeability having the dimension of velocity;

m = slope of the linear logarithmic equation representing the free surface near the well;

p = unit pressure;

q = discharge between flow surfaces;

Q = total discharge;

r = radial distance from axis of well:

- r_e = radius of influence;
- r_w = radius of the well;

s = length along a stream line;

t = thickness;

v = velocity in Darcy's equation;

z = elevation; and

γ = weight per unit volume.